3-3 Study Guide and Intervention

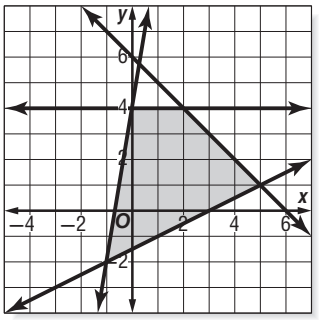
Optimization with Linear Programming

Maximum and Minimum Values When a system of linear inequalities produces a bounded polygonal region, the *maximum* or *minimum* value of a related function will occur at a vertex of the region.

Example **Graph the system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the function $f(x, y) = 3x + 2y$ for this polygonal region.**

$$y \le 4$$
$$y \le -x + 6$$
$$y \ge \frac{1}{2}x - \frac{3}{2}$$
$$y \le 6x + 4$$

First find the vertices of the bounded region. Graph the inequalities.



The polygon formed is a quadrilateral with vertices at (0, 4), (2, 4), (5, 1), and (−1, −2). Use the table to find the maximum and minimum values of $f(x, y) = 3x + 2y$.

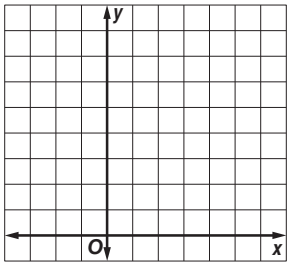
(x, y)	$3x + 2y$	$f(x, y)$
(0, 4)	3(0) + 2(4)	8
(2, 4)	3(2) + 2(4)	14
(5, 1)	3(5) + 2(1)	17
(−1, −2)	3(−1) + 2(−2)	−7

The maximum value is 17 at (5, 1). The minimum value is −7 at (−1, −2).

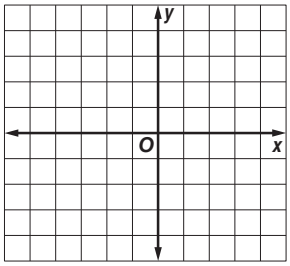
Exercises

Graph each system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the given function for this region.

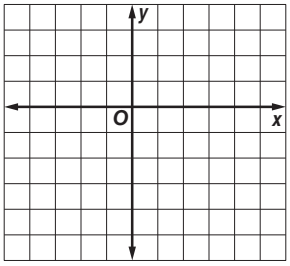
1. $y \ge 2$
$1 \le x \le 5$
$y \le x + 3$
$f(x, y) = 3x - 2y$



2. $y \ge -2$
$y \ge 2x - 4$
$x - 2y \ge -1$
$f(x, y) = 4x - y$



3. $x + y \ge 2$
$4y \le x + 8$
$y \ge 2x - 5$
$f(x, y) = 4x + 3y$



3-3 Study Guide and Intervention *(continued)*

Optimization with Linear Programming

Optimization When solving **linear programming** problems, use the following procedure.

1. Define variables.
2. Write a system of inequalities.
3. Graph the system of inequalities.
4. Find the coordinates of the vertices of the feasible region.
5. Write an expression to be maximized or minimized.
6. Substitute the coordinates of the vertices in the expression.
7. Select the greatest or least result to answer the problem.

Example **A painter has exactly 32 units of yellow dye and 54 units of green dye. He plans to mix as many gallons as possible of color A and color B. Each gallon of color A requires 4 units of yellow dye and 1 unit of green dye. Each gallon of color B requires 1 unit of yellow dye and 6 units of green dye. Find the maximum number of gallons he can mix.**

Step 1 Define the variables.

x = the number of gallons of color A made

y = the number of gallons of color B made

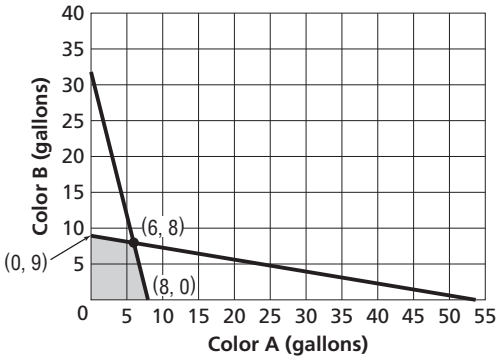
Step 2 Write a system of inequalities.

Since the number of gallons made cannot be negative, $x \geq 0$ and $y \geq 0$.

There are 32 units of yellow dye; each gallon of color A requires 4 units, and each gallon of color B requires 1 unit.

So $4x + y \leq 32$.

Similarly for the green dye, $x + 6y \leq 54$.



Steps 3 and 4 Graph the system of inequalities and find the coordinates of the vertices of the feasible region. The vertices of the feasible region are (0, 0), (0, 9), (6, 8), and (8, 0).

Steps 5–7 Find the maximum number of gallons, $x + y$, that he can make. The maximum number of gallons the painter can make is 14, 6 gallons of color A and 8 gallons of color B.

Exercises

1. **FOOD** A delicatessen has 12 pounds of plain sausage and 10 pounds of spicy sausage. A pound of Bratwurst A contains $\frac{3}{4}$ pound of plain sausage and $\frac{1}{4}$ pound of spicy sausage. A pound of Bratwurst B contains $\frac{1}{2}$ pound of each sausage. Find the maximum number of pounds of bratwurst that can be made.

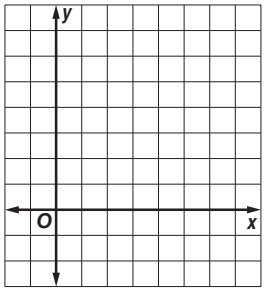
2. **MANUFACTURING** Machine A can produce 30 steering wheels per hour at a cost of \$8 per hour. Machine B can produce 40 steering wheels per hour at a cost of \$12 per hour. The company can use either machine by itself or both machines at the same time. What is the minimum number of hours needed to produce 380 steering wheels if the cost must be no more than \$108?

3-3 Skills Practice

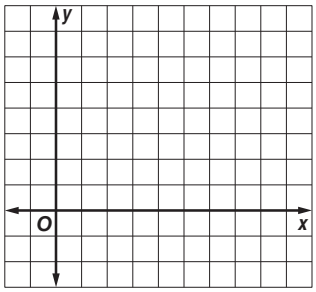
Optimization with Linear Programming

Graph each system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the given function for this region.

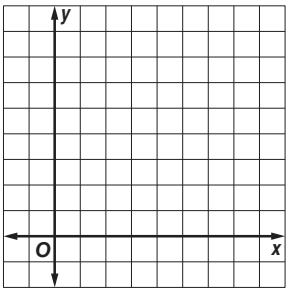
1. $x \geq 2$
$x \leq 5$
$y \geq 1$
$y \leq 4$
$f(x, y) = x + y$



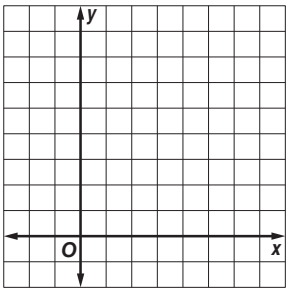
2. $x \geq 1$
$y \leq 6$
$y \geq x - 2$
$f(x, y) = x - y$



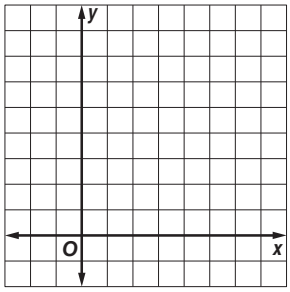
3. $x \geq 0$
$y \geq 0$
$y \leq 7 - x$
$f(x, y) = 3x + y$



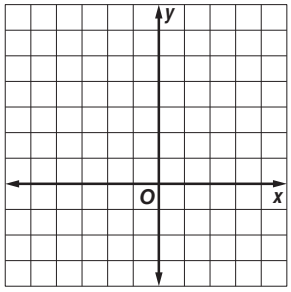
4. $x \geq -1$
$x + y \leq 6$
$f(x, y) = x + 2y$



5. $y \leq 2x$
$y \geq 6 - x$
$y \leq 6$
$f(x, y) = 4x + 3y$



6. $y \geq -x - 2$
$y \geq 3x + 2$
$y \leq x + 4$
$f(x, y) = -3x + 5y$



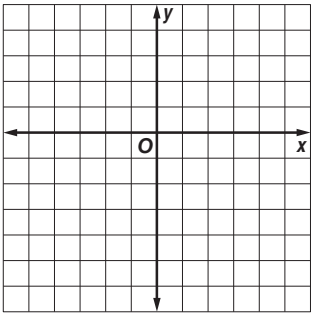
7. MANUFACTURING A backpack manufacturer produces an internal frame pack and an external frame pack. Let x represent the number of internal frame packs produced in one hour and let y represent the number of external frame packs produced in one hour. Then the inequalities $x + 3y \leq 18$, $2x + y \leq 16$, $x \geq 0$, and $y \geq 0$ describe the constraints for manufacturing both packs. Use the profit function $f(x, y) = 50x + 80y$ and the constraints given to determine the maximum profit for manufacturing both backpacks for the given constraints.

3-3 Practice

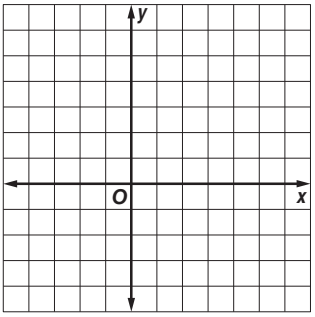
Optimization with Linear Programming

Graph each system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the given function for this region.

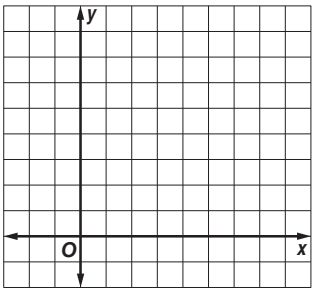
1. $2x - 4 \le y$
$-2x - 4 \le y$
$y \le 2$
$f(x, y) = -2x + y$



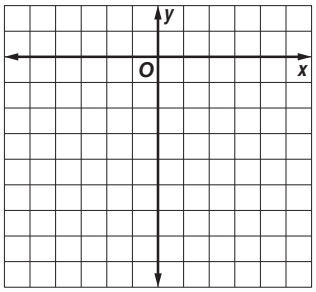
2. $3x - y \le 7$
$2x - y \ge 3$
$y \ge x - 3$
$f(x, y) = x - 4y$



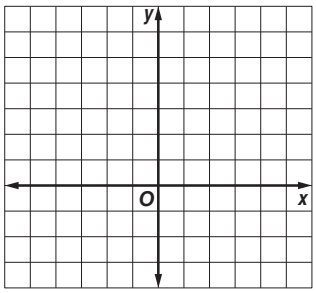
3. $x \ge 0$
$y \ge 0$
$y \le 6$
$y \le -3x + 15$
$f(x, y) = 3x + y$



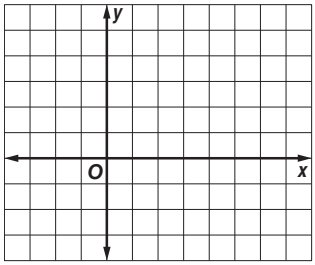
4. $x \le 0$
$y \le 0$
$4x + y \ge -7$
$f(x, y) = -x - 4y$



5. $y \le 3x + 6$
$4y + 3x \le 3$
$x \ge -2$
$f(x, y) = -x + 3y$



6. $2x + 3y \ge 6$
$2x - y \le 2$
$x \ge 0$
$y \ge 0$
$f(x, y) = x + 4y + 3$



7. PRODUCTION A glass blower can form 8 simple vases or 2 elaborate vases in an hour. In a work shift of no more than 8 hours, the worker must form at least 40 vases.

a. Let s represent the hours forming simple vases and e the hours forming elaborate vases. Write a system of inequalities involving the time spent on each type of vase.

b. If the glass blower makes a profit of \$30 per hour worked on the simple vases and \$35 per hour worked on the elaborate vases, write a function for the total profit on the vases.

c. Find the number of hours the worker should spend on each type of vase to maximize profit. What is that profit?

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3-3 Word Problem Practice

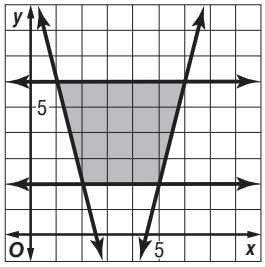
Optimization with Linear Programming

1. REGIONS A region in the plane is formed by the equations $x - y < 3$, $x - y > -3$, and $x + y > -3$. Is this region bounded or unbounded? Explain.

2. MANUFACTURING Eighty workers are available to assemble tables and chairs. It takes 5 people to assemble a table and 3 people to assemble a chair. The workers always make at least as many tables as chairs because the tables are easier to make. If x is the number of tables and y is the number of chairs, the system of inequalities that represent what can be assembled is $x > 0$, $y > 0$, $y \leq x$, and $5x + 3y \leq 80$. What is the maximum total number of chairs and tables the workers can make?

3. FISH An aquarium is 7000 cubic inches. Nathan wants to populate the aquarium with neon tetras and catfish. It is recommended that each neon tetra be allowed 170 cubic inches and each catfish be allowed 700 cubic inches of space. Nathan would like at least one catfish for every 4 neon tetras. Let n be the number of neon tetra and c be the number of catfish. The following inequalities form the feasible region for this situation: $n > 0$, $c > 0$, $4c \geq n$, and $170n + 700c \leq 7000$. What is the maximum number of fish Nathan can put in his aquarium?

4. ELEVATION A trapezoidal park is built on a slight incline. The function for the ground elevation above sea level is $f(x, y) = x - 3y + 20$ feet. What are the coordinates of the highest point in the park?



5. CERAMICS Josh has 8 days to make pots and plates to sell at a local fair. Each pot weighs 2 pounds and each plate weighs 1 pound. Josh cannot carry more than 50 pounds to the fair. Each day, he can make at most 5 plates and at most 3 pots. He will make \$12 profit for every plate and \$25 profit for every pot that he sells.

a. Write linear inequalities to represent the number of pots p and plates a Josh may bring to the fair.

b. List the coordinates of the vertices of the feasible region.

c. How many pots and how many plates should Josh make to maximize his potential profit?

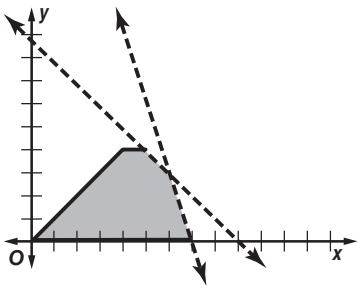
3-3 Enrichment

Sensitivity Analysis

A linear programming model has specific objective coefficients. For example, if the value of a model is found by $2x + 3y = 5$, the objective coefficients are {2, 3}. What if these coefficients were {2.1, 2.9} or {2.5, 3.1}? How would these changes affect the optimal linear program value? This type of investigation is called **sensitivity analysis**.

In general, the objective function in two-variable linear programming problem can be written as: maximize (or minimize) $Ax + By = C$, subject to a set of constraint equations. Changes to the *parameters* A and B could change the slope of the line. This change of slope could lead to a change in the optimum solution to a different corner point (Recall, the optimum solution occurs at a *corner point*).

There is a range in the slope value that will produce this change, thus there is a range of variation for both A and B that will keep the optimal solution the same (see graph).



1. Find the slope of $Ax + By = C$ and observe how changes to the parameters A and B can change the slope of the line.

Consider the Linear Programming problem:

Maximize: $C = 2x + 3y$,
Subject to: $3x + y \leq 21$
$x + y \leq 9$
$y \leq x$
$y \leq 4$

(x, y)	(0, 0)	(4, 4)	(5, 4)	(6, 3)	(7, 0)
C	0	20	22	21	14

After finding the intersections and evaluating the objective equation, we find the maximal solution is (5, 4). If the objective coefficients are changed from 2 and 3 to A and B, the optimum solution with remain at (5, 4) while the slope remains between the slope of $x + y \leq 9$ and the slope of $3x + y \leq 21$. If not, then the new optimal solution will be at (4, 4) or (6, 3).

2. Express the relationship, the slope of the objective function is between the slope of the line $x + y = 9$ and the slope of the line $3x + y = 21$, algebraically.

3. Determine the range on A if B remains equal to 3.

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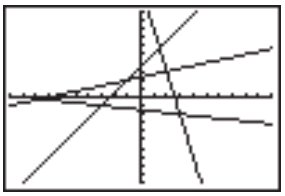
3-3 Graphing Calculator Activity

Optimization with Linear Programming

A graphing calculator can store the x- and y-coordinates when using the **intersect** command in the **[CALC]** menu. These values can be used to find the vertices of the feasible region and determine the maximum or minimum value for $f(x, y)$.

Example **Graph the system $x - 3y \geq -7$, $5x + y \geq 13$, $x + 6y \geq -9$, $3x - 2y \geq -7$, and $f(x, y) = 4x - 3y$. Find the coordinates of the feasible region. Then find the maximum and minimum values for the system.**

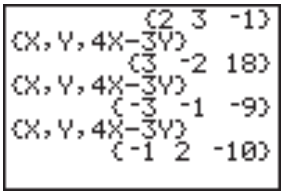
Solve each inequality for y. Enter each boundary equation in the **Y=** screen.



[−10, 10] scl:1 by [−10, 10] scl:1

Step 1: Enter the equations of the boundaries.

Keystrokes: [Y=] [(] 1 [÷] 3 [)] [X,T,Θ,n] [+] [(] 7 [÷] 3 [)] [ENTER] [(–)] 5 [X,T,Θ,n] [+] 13 [ENTER] [(–)] [(] 1 [÷] 6 [)] [X,T,Θ,n] [—] [(] 3 [÷] 2 [)] [ENTER] [(] 3 [÷] 2 [)] [X,T,Θ,n] [+] [(] 7 [÷] 2 [)] [ZOOM] 6



Step 2: Find the vertices of the region.

Keystrokes: [2nd] [CALC] 5 [ENTER] [ENTER] [ENTER] [2nd] [QUIT]

Step 3: Enter the function.

Keystrokes: [2nd] [{] [X,T,Θ,n] [,] [ALPHA] [Y] [,] 4 [X,T,Θ,n] [—] 3 [ALPHA] [Y] [2nd] [}] [ENTER]

Step 4: Calculate the values.

Keystrokes: [GRAPH] [2nd] [CALC] 5 [▼] [ENTER] [ENTER] [ENTER] [2nd] [QUIT] [2nd] [ENTRY] [ENTER] [GRAPH] [2nd] [CALC] 5 [▼] [▼] [ENTER] [ENTER] [ENTER] [2nd] [QUIT] [2nd] [ENTRY] [ENTER] [GRAPH] [2nd] [CALC] 5 [ENTER] [▲] [▲] [ENTER] [ENTER] [2nd] [QUIT] [2nd] [ENTRY] [ENTER].

The maximum value of the system is 18 and the minimum value is −10.

Exercises

Graph each system. Find the coordinates of the vertices of the feasible region. Then find the maximum and minimum values for the system.

1. $2x + 3y \geq 6$
$3x - 2y \geq -4$
$5x + y \geq 15$
$f(x, y) = x + 3y$

2. $y \leq 4x + 6$
$x + 4y \leq 7$
$2x + y \leq 7$
$x - 6y \leq 10$
$f(x, y) = 2x - y$

3. $y \leq 16 - x$
$0 \leq 2y \leq 7$
$2x + 3y \geq 11$
$y \leq 3x + 1$
$y \geq 2x - 13$
$y \geq 7 - 2x$
$f(x, y) = 5x + 6y$